

Beyond the Schwinger boson representation of the $su(2)$ -algebra. I

— *New boson representation based on the $su(1,1)$ -algebra and its related problems with application* —

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With the use of two kinds of boson operators, a new boson representation of the $su(2)$ -algebra is proposed. The basic idea comes from the pseudo $su(1,1)$ -algebra recently given by the present authors. It forms a striking contrast to the Schwinger boson representation of the $su(2)$ -algebra which is also based on two kinds of bosons. This representation may be suitable for describing time-dependence of the system interacting with the external environment in the framework of the thermo field dynamics formalism, i.e., the phase space doubling. Further, several deformations related to the $su(2)$ -algebra in this boson representation are discussed. On the basis of these deformed algebra, various types of time-evolution of a simple boson system are investigated.

§1. Introduction

It may be hardly necessary to mention, but the $su(2)$ -algebra has made a central contribution to the development of microscopic study of nuclear structure. The BCS-Bogoliubov theory may be its typical example. We know that the $su(2)$ -algebra may be the simplest and the most popular Lie algebra in nuclear structure theory. In many cases, the Lie algebras have been treated under the name of boson realizations of Lie algebras. The prototype has been called as boson expansion or boson mapping for many-fermion systems, which is related to the $so(2N)$ -algebra for a space of N single-particle states.¹⁾ With the development of the study of the boson expansion, it became to be called the boson realization of the Lie algebra.²⁾ The simplest case is the boson realization of the $su(2)$ -algebra, which is called as the Holstein-Primakoff representation.³⁾ The three generators are expressed in terms of one kind of boson operator for a given value of the magnitude of the $su(2)$ -spin, which determines the irreducible representation. Of course, it is positive-definite. We know also another boson representation of the $su(2)$ -algebra, that is, the Schwinger boson representation.⁴⁾ The explicit form will be given in the relation (3.1) and (3.2). It consists of two kinds of boson operators. Different from the Holstein-Primakoff representation, the magnitude of the $su(2)$ -spin is expressed in terms of an operator, i.e., half of the sum of two boson number operators, which can be seen in the relation (3.2b). Clearly, it is a positive-definite operator. On the other hand, we know the boson representation of the $su(1,1)$ -algebra initiated also by Schwinger.⁴⁾ This representation is also constructed in terms of two kinds of boson operators. The explicit

form will be presented in the relations (2.1) and (2.3). In contrast with the case of the $su(2)$ -algebra, in this representation, the quantum numbers specifying the irreducible representation are the eigenvalues of an operator which is related to half of the difference of the two boson number operators, which can be seen in the relation (2.3b). Therefore, this operator is not positive-definite. On the analogy with the magnitude of the $su(2)$ -spin, we will call the positive eigenvalue as the magnitude of the $su(1,1)$ -spin.

The Schwinger boson representation of the $su(1,1)$ -algebra may be not familiar to the field of nuclear structure theory mainly treating the zero temperature. A merit of this representation is to be able to describe damping and amplifying of an isolated harmonic oscillator induced classically mechanically by the velocity-dependent force quantum mechanically in conservative form.^{5),6),7)} In the background of this description, there exists the idea of the thermo field dynamics formalism based on the phase space doubling.⁸⁾ We should note the following: The above-mentioned isolated oscillator is of one dimension and the Schwinger boson representation is constructed in two dimensional space, because of the use of two kinds of bosons. Therefore, the idea of the phase space doubling is conjectured to be useful in the present problem. The original intrinsic oscillator is isolated and it is expressed in terms of one kind of boson and external environment in terms of another kind of boson, in which the frequency is the same as that of the original one. The interaction between both systems is naturally introduced.

If we follow the idea of the phase space doubling, the total Hamiltonian for the present system, \hat{H} , can be expressed in the form

$$\hat{H} = \hat{H}_0 + \hat{V}_i, \quad (1.1a)$$

$$\hat{H}_0 = \hat{H}_{\text{intr}} - \hat{H}_{\text{extr}}. \quad (1.1b)$$

Here, \hat{H}_{intr} and \hat{H}_{extr} denote the Hamiltonians of the intrinsic and the external environment system, respectively. Both Hamiltonians are of the harmonic oscillator type with the same frequencies. The part \hat{V}_i denotes the interaction between both systems. It should be noted that \hat{H} is not the total energy operator but the operator for time-evolution of the system. As for \hat{V}_i , a certain linear combination of the raising and the lowering operator of the $su(1,1)$ -algebra in the Schwinger boson representation was adopted in Refs.5),6),7). Further, we notice that \hat{H}_0 is a constant of motion, because \hat{H}_0 is related to the magnitude of the $su(1,1)$ -spin. In Refs.6),7), the Hamiltonian (1.1) was treated in the framework of the time-dependent variational method. Of course, under a careful consideration on the magnitude of the $su(1,1)$ -spin as a constant of motion in the $su(1,1)$ -algebra, the trial state is constructed. Therefore, the expectation value of \hat{H}_0 does not depend on time, but the expectation value of \hat{H}_{intr} depends on time. Under the use of \hat{V}_i mentioned above, the expectation value becomes decreasing or increasing function of time. The former and the latter show the damped and the amplified oscillation, respectively. Some variations of the above-mentioned idea were discussed in Ref.7).

However, we must point out that the $su(1,1)$ -algebra in the Schwinger boson representation cannot be applied directly to many-fermion system. The reason is very simple: We cannot find the $su(1,1)$ -algebra in many-fermion system. In response

to this situation, the present authors proposed an idea.⁹⁾ Hereafter, Ref.9) will be referred to as (A). The orthogonal set constructed under the Schwinger boson representation of the $su(1,1)$ -algebra forms an infinite dimensional space for a given value of the magnitude of the $su(1,1)$ -spin. But, the orthogonal set for many-fermion system is of finite dimension for a given value of the magnitude of the $su(2)$ -spin. Then, a possible idea is to define, in the infinite dimensional space, a certain subspace which is in one-to-one correspondence to the fermion space. Further, we required that any matrix element of the raising and the lowering operator in this subspace does not change the form from that given in this algebra. If restricted to this subspace, the algebra becomes deformed from the $su(1,1)$ -algebra. In Ref.9), we call it the pseudo $su(1,1)$ -algebra. Three generators are expressed in terms of the original $su(1,1)$ -generators with certain parameter which is closely related with the dimension of the subspace. Then, we can connect the pseudo $su(1,1)$ -algebra with the $su(2)$ -algebra, for example, which governs the Cooper pair. The above idea suggests us that many-fermion system can be treated by the pseudo $su(1,1)$ -algebra and in (A), we described a simple fermion system based on the thermo field dynamics formalism. As a result, the periodical dependence of the energy of the intrinsic system on time was shown. This result is in contrast to that in the $su(1,1)$ -algebra. For the time-dependent variational method adopted in (A), we prepare a trial state which contains one complex parameter for the variation and the normalization constant of the trial state must be given. But, in the case of the pseudo $su(1,1)$ -algebra, the explicit form of the normalization constant is too complicated to treat it practically. It was shown in (A).

Main aim of this paper is to present a new boson representation of the $su(2)$ -algebra which may be suitable for the application of the idea of the phase space doubling. As was already mentioned, the Schwinger representation is powerless to the use of the phase space doubling. In contrast to the case of the Schwinger boson representation, the operator for the magnitude of the $su(2)$ -spin in the present case is expressed in terms of the form related to the difference of two boson number operators. Under an idea similar to that for constructing the pseudo $su(1,1)$ -algebra, we can express the $su(2)$ -generators as functions of the three $su(1,1)$ -generators and a certain parameter. If the value of this parameter is appropriately chosen, the present representation is reduced to the boson realization of the $su(2)$ -algebra governing, for example, the Cooper pair. Of course, these generators are available in the subspace leading the pseudo $su(1,1)$ -algebra. In Part II, we will give the proof. We intend to apply the present representation to the Hamiltonian (1.1) under the same manner as that in (A). Interaction term \hat{V}_i must be appropriately chosen. Therefore, we prepare a certain trial state for the time-dependent variation so as to calculate the normalization constant easily. Expecting various results, we give several types of the deformations from the new boson representation of the $su(2)$ -algebra. As was already mentioned, the algebra discussed in this paper is applied to the Hamiltonian (1.1). As for \hat{V}_i , we adopt a certain linear combination of the raising and the lowering operator of the algebra obtained by each deformation and we can show that the expectation value of the intrinsic Hamiltonian, \hat{H}_{intr} , changes periodically for time. Depending on the choice of the deformation, the features of the change show various

shapes. In classical mechanics, we know one problem: Elastic collision of simply oscillating light particle with the sufficient heavy one on the horizontal straight line. We can show that the problem discussed in this paper reduces to the same as the above.

After, in §2, recapitulating the pseudo $su(1,1)$ -algebra presented in (A) with some new aspects, a new boson representation of the $su(2)$ -algebra is proposed in §3. It is based on the Schwinger boson representation of the $su(1,1)$ -algebra. The proof is given in Part II. Section 4 is devoted to giving various deformation from the new boson representation. Normalization constant in the orthogonal set obtained in each deformation is easily calculated with rather simple approximation. In §§5 and 6, the time-evolution is investigated for the boson system characterized by the Hamiltonian (1.1). Depending chosen deformation, the change of the energy of the intrinsic system for time is periodic, but, the behaviors are different from one another.

§2. Pseudo $su(1,1)$ -algebra in the Schwinger boson representation

Our preliminary task is to recapitulate the Schwinger boson representation of the $su(1,1)$ -algebra in a form suitable for later discussion. The details can be seen in Refs.(4), (5), (6), (7). It starts in the following three operators:

$$\hat{T}_+ = \hat{a}^* \hat{b}^* , \quad \hat{T}_- = \hat{b} \hat{a} , \quad \hat{T}_0 = \frac{1}{2}(\hat{a}^* \hat{a} + \hat{b}^* \hat{b} + 1) . \quad (2.1)$$

Here, (\hat{a}, \hat{a}^*) and (\hat{b}, \hat{b}^*) denote two kinds of the boson operators. The operators $\hat{T}_{\pm,0}$ form the $su(1,1)$ -algebra:

$$\hat{T}_-^* = \hat{T}_+ , \quad \hat{T}_0^* = \hat{T}_0 , \quad (2.2a)$$

$$[\hat{T}_+ , \hat{T}_-] = -2\hat{T}_0 , \quad [\hat{T}_0 , \hat{T}_{\pm}] = \pm \hat{T}_{\pm} . \quad (2.2b)$$

The Casimir operator $\hat{\mathbf{T}}^2$, together with its properties, is given by

$$\hat{\mathbf{T}}^2 = \hat{T}_0^2 - \frac{1}{2}(\hat{T}_- \hat{T}_+ + \hat{T}_+ \hat{T}_-) = \hat{T}(\hat{T} - 1) , \quad (2.3a)$$

$$\hat{T} = \frac{1}{2}(-\hat{a}^* \hat{a} + \hat{b}^* \hat{b} + 1) , \quad [\hat{T} , \hat{T}_{\pm,0}] = 0 . \quad (2.3b)$$

The eigenstate of \hat{T} and \hat{T}_0 with its eigenvalue t and t_0 , respectively, is obtained in the form

$$|t, t_0\rangle = \frac{1}{\sqrt{(t_0 - t)!(t_0 + t - 1)!}} (\hat{a}^*)^{t_0 - t} (\hat{b}^*)^{t_0 + t - 1} |0\rangle , \quad (\text{normalized})$$

$$\hat{a}|0\rangle = \hat{b}|0\rangle = 0 . \quad (2.4)$$

Since $t_0 - t = 0, 1, 2, 3, \dots$ and $t_0 + t - 1 = 0, 1, 2, 3, \dots$, $|t, t_0\rangle$ is treated separately in the following two cases:

$$(i) \quad t = 1/2, 1, 3/2, \dots, \infty , \quad t_0 = t, t+1, t+2, \dots, \infty \quad \text{for a given } t , \quad (2.5a)$$

$$(ii) \quad t = 0, -1/2, -1, \dots, -\infty , \quad t_0 = -t+1, -t+2, -t+3, \dots, \infty \quad \text{for a given } t . \quad (2.5b)$$

It is noted that, in this paper, only the case (i) will be treated. In Part II, we will contact with the case (ii) briefly in connection with the case (i). The state $|t, t_0\rangle$ can be rewritten in the form

$$|t, t_0\rangle = \frac{1}{\sqrt{(t_0 - t)!(t_0 + t - 1)!}} \left(\hat{T}_+\right)^{t_0 - t} |t\rangle . \quad (|t, t_0 = t\rangle = |t\rangle) \quad (2.6a)$$

$$\text{i.e. } |n; t\rangle = \frac{1}{\sqrt{n!(2t - 1 + n)!}} \left(\hat{T}_+\right)^n |t\rangle . \quad (2.6b)$$

Here, $|t\rangle$ denotes the minimum weight state:

$$|t\rangle = (\hat{b}^*)^{2t-1} |0\rangle , \quad \hat{T}_- |t\rangle = 0 , \quad \hat{T}_0 |t\rangle = \hat{T} |t\rangle = t |t\rangle . \quad (2.7)$$

By n -time operation of the raising operator \hat{T}_+ on $|t\rangle$, $|n; t\rangle$ is obtained. It should be noted that this operation is permitted until the infinite time, in other word, there does not exist the maximum weight state. Clearly, {the states (2.6)} forms an orthogonal set with the infinite dimension.

Recently, a possible form of the pseudo $su(1, 1)$ -algebra was presented by the present authors.⁹⁾ It aims at demonstrating a deformation of the Cooper pair obeying the $su(2)$ -algebra in many-fermion system. Hereafter, it will be referred to as (A). In (A), one type of possible deformations of the $su(1, 1)$ -algebra in the Schwinger boson representation was treated and we called it the pseudo $su(1, 1)$ -algebra. In this paper, this algebra is formulated in a manner slightly modified form that given in (A). Basic scheme of the pseudo $su(1, 1)$ -algebra is to construct it in the subspace of the space (2.5a):

$$t = 1/2, 1, 3/2, \dots, \mu - 1/2, \mu, \quad t_0 = t, t + 1, t + 2, \dots, t_m - 1, t_m \quad \text{for a given } t . \quad (2.8)$$

Here, μ and t_m denote integer or half-integer, where t_m is a function of t . Depending on the model under investigation, the value of μ and the functional form of t_m are chosen appropriately. Later, we will show a possible idea for the determination of μ and t_m .

Let $\hat{T}_{\pm, 0}$ denote three generators of the pseudo $su(1, 1)$ -algebra. Role of $\hat{T}_{\pm, 0}$ is the same as that of $\hat{T}_{\pm, 0}$. One time operation of \hat{T}_+ makes the eigenvalue of \hat{T}_0 increase by one in the state specified by the quantum number (t, t_0) . In this algebra, there exist not only the minimum but also the maximum weight state. Further, the following is required for a given t : the minimum and the maximum weight state are identical to $|t, t\rangle (= |t\rangle)$ and $|t, t_m\rangle$, respectively, and successive operation of \hat{T}_+ on $|t\rangle$ reduces to the state (2.6). The above requirement is formulated as follows:

$$\hat{T}_- |t, t\rangle = 0 , \quad \hat{T}_0 |t, t\rangle = t |t, t\rangle , \quad (2.9a)$$

$$\hat{T}_+ |t, t_m\rangle = 0 , \quad \hat{T}_0 |t, t_m\rangle = t_m |t, t_m\rangle , \quad (2.9b)$$

$$\left(\hat{T}_+\right)^{t_0 - t} |t\rangle = \left(\hat{T}_+\right)^{t_0 - t} |t\rangle . \quad (2.9c)$$

As was already mentioned, the eigenvalue of \hat{T}_0 increases one by one from t to t_m and, then, \hat{T}_0 may be permitted to be of the form

$$\hat{T}_0 = \hat{T} + f(\hat{T}) . \quad (2.10)$$

The reason is very simple. The eigenvalue of \hat{T}_0 increases one by one and $f(\hat{T})$ does not make the eigenvalue of \hat{T}_0 change. Then, by operating $\hat{\mathcal{T}}_0$ on the states $|t, t\rangle$ and $|t, t_m\rangle$, the following relation is obtained:

$$t + f(t) = t, \quad t_m + f(t) = t_m, \quad \text{i.e.,} \quad f(t) = 0. \quad (2.11)$$

Therefore, $\hat{\mathcal{T}}_0$ is equal to \hat{T}_0 :

$$\hat{\mathcal{T}}_0 = \hat{T}_0. \quad (2.12a)$$

If noticing the relation $\hat{T}_-|t\rangle = 0$ and $\sqrt{t_m - \hat{T}_0} \cdot (\sqrt{t_m - \hat{T}_0} + \epsilon)^{-1}|t, t_m\rangle = \sqrt{0}/\sqrt{\epsilon}|t, t_m\rangle \rightarrow 0$ ($\epsilon \rightarrow 0$), the requirement (2.9) suggests us the following form for $\hat{\mathcal{T}}_{\pm}$:

$$\hat{\mathcal{T}}_+ = \hat{T}_+ \cdot \sqrt{t_m - \hat{T}_0} \cdot \left(\sqrt{t_m - \hat{T}_0} + \epsilon \right)^{-1}, \quad \hat{\mathcal{T}}_- = \left(\sqrt{t_m - \hat{T}_0} + \epsilon \right)^{-1} \cdot \sqrt{t_m - \hat{T}_0} \cdot \hat{T}_-. \quad (2.12b)$$

Here, as was already shown, ϵ denotes positive infinitesimal parameter. With the use of the commutation relation (2.2b), the arrangement of the operators can be changed.

It was already mentioned that t_m is a function of t , i.e., $t_m = F_m(t)$. Depending on the problem under investigation, the form of $F_m(t)$ is fixed. If intending to apply the present algebra to the investigation of such state as boson coherent state, the present one must be formulated in the operator form, that is, there do not appear such quanta as t_0 , t and t_m . Then, it may be permissible to define an operator $\hat{T}_m = F_m(\hat{T})$ and if \hat{T} is replaced with t in $F_m(\hat{T})$, \hat{T}_m becomes $F_m(t) = t_m$. Therefore, $\hat{\mathcal{T}}_{\pm,0}$ can be expressed as

$$\begin{aligned} \hat{\mathcal{T}}_+ &= \hat{T}_+ \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \left(\sqrt{\hat{T}_m - \hat{T}_0} + \epsilon \right)^{-1}, \\ \hat{\mathcal{T}}_- &= \left(\sqrt{\hat{T}_m - \hat{T}_0} + \epsilon \right)^{-1} \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \hat{T}_-, \\ \hat{\mathcal{T}}_0 &= \hat{T}_0. \end{aligned} \quad (2.13)$$

The operator $\hat{\mathcal{T}}_{\pm,0}$ satisfy

$$\begin{aligned} \hat{\mathcal{T}}_-^* &= \hat{\mathcal{T}}_+, \quad \hat{\mathcal{T}}_0^* = \hat{\mathcal{T}}_0, \\ [\hat{\mathcal{T}}_+, \hat{\mathcal{T}}_-] &= -2\hat{\mathcal{T}}_0 + \epsilon \frac{(\hat{T}_0 + \hat{T})(\hat{T}_0 - \hat{T} + 1)}{\hat{T}_m - \hat{T}_0 + \epsilon}, \end{aligned} \quad (2.14a)$$

$$[\hat{\mathcal{T}}_0, \hat{\mathcal{T}}_{\pm}] = \pm \hat{\mathcal{T}}_{\pm}. \quad (2.14b)$$

The operator $\hat{\mathcal{T}}^2$ corresponding to the Casimir operator \hat{T}^2 is obtained in the form

$$\begin{aligned} \hat{\mathcal{T}}^2 &= \hat{\mathcal{T}}_0^2 - \frac{1}{2} (\hat{\mathcal{T}}_- \hat{\mathcal{T}}_+ + \hat{\mathcal{T}}_+ \hat{\mathcal{T}}_-) \\ &= \hat{\mathcal{T}} (\hat{\mathcal{T}} - 1) + \frac{1}{2} \epsilon \frac{(\hat{T}_0 + \hat{T})(\hat{T}_0 - \hat{T} + 1)}{\hat{T}_m - \hat{T}_0 + \epsilon}, \quad \hat{\mathcal{T}} = \hat{T}. \end{aligned} \quad (2.15)$$

If, in second terms on the right-hand sides of the relations (2.14a) and (2.15) the limiting process $\hat{T}_m \rightarrow \infty$ proceeds to replacing \hat{T}_0 and \hat{T} with the eigenvalues t_0 and t , the above algebra reduces to the $su(1, 1)$ -algebra. This can be seen in the form of the generators (2.12) directly. For example, in the case $\hat{T}_m = C_m + 1 - \hat{T}$, $\hat{T}_m \rightarrow \infty$ if $C_m \rightarrow \infty$. Of course, C_m is a parameter. This formula may be understood in the following manner: In the state (2.9c) for a given t , the created bosons consist of $2(t_m - t)$ ($= 2(F_m(t) - t)$) bosons in the a - b pair type and $(2t - 1)$ bosons in the single b -boson type. In the present scheme, the boson in the single a -boson type is forbidden to create. In the case $t = 1/2$, the created bosons are all in the a - b pair type, i.e., the number of the created bosons is $2(F_m(1/2) - 1/2)$. The above consideration suggests us the following aspect: In the state (2.9c), there exist $(2t - 1)$ vacancies and if $(2t - 1)$ a -bosons occupy these vacancies, all the created bosons come to be in the a - b pair type. On the basis of the above argument, we require that independent of the value of t , the number of the created bosons is limited in the present system. Then, the following relation may be acceptable:

$$2(F_m(t) - t) + 2(2t - 1) = 2 \left(F_m \left(\frac{1}{2} \right) - \frac{1}{2} \right) (= 2C_m) ,$$

i.e., $t_m = C_m + 1 - t$. (2.16a)

Since $t_m = F_m(t)$ is decreasing for t , there exists a certain point $t = t_c$ satisfying $F_m(t_c) = t_c$ and, then, t_c is nothing but μ ($t_c = \mu$). This argument gives us

$$\mu = \frac{1}{2}(C_m + 1) . \quad (2.16b)$$

With the aid of the $su(1, 1)$ - and the pseudo $su(1, 1)$ -algebra in the Schwinger boson representation, the role of the phase space doubling mentioned qualitatively in §1 can be understood rather quantitatively. From the relations (2.1) and (2.3b), the following expression for $\hat{b}^* \hat{b}$ is derived:

$$\hat{b}^* \hat{b} = \hat{T}_0 + \hat{T} - 1 . \quad (2.17a)$$

Let a time-dependent state vector, which is the eigenstate of \hat{T} with the eigenvalue t , exist. This is of the superposition of the orthogonal set (2.4) or (2.6) from $t_0 = t$ to $t \rightarrow \infty$. Then, the expectation value of $\hat{b}^* \hat{b}$ at the time τ is expressed as

$$\langle \hat{b}^* \hat{b} \rangle_\tau = t_0(\tau) + t - 1 , \quad t_0(\tau) = \langle \hat{T}_0 \rangle_\tau , \quad t_0(0) > t . \quad (2.17b)$$

The damped and amplified harmonic oscillator can be understood by investigating the behavior of $t_0(\tau)$, which is determined by the Hamiltonian adopted in the model under investigation. If $t_0(\tau)$ is monotone-decreasing ($t_0(0) > t_0(\tau) > t$), $t_0(\tau) \rightarrow t$ ($\tau \rightarrow \infty$). Then, $\langle \hat{H}_{\text{intr}} \rangle_\tau = \langle \omega \hat{b}^* \hat{b} \rangle_\tau$ decreases from the value $\omega(t_0(\tau) + t - 1)$ to $\omega(2t - 1)$ at the limit $\tau \rightarrow \infty$. This corresponds to the damped oscillator. On the other hand, if $t_0(\tau)$ is monotone-increasing ($t_0(0) < t_0(\tau)$), $t_0(\tau) \rightarrow \infty$ ($\tau \rightarrow \infty$). Then, $\langle \omega \hat{b}^* \hat{b} \rangle_\tau$ increases from the value $\omega(t_0(0) + t - 1)$ to ∞ at the time $\tau \rightarrow \infty$. This case corresponds to the amplified oscillation.

It may be interesting to investigate the pseudo $su(1,1)$ -algebra in relation to the boson realization of many-fermion system. In (A), a possible deformation of the Cooper pair in the frame of this algebra was discussed. Let a time-dependent state, which is the eigenstate of \hat{T} with the eigenvalue t , exist. In this case, the state is of a superposition of the eigenstates of \hat{T}_0 , the eigenvalues of which change from $t_0 = t$ to $t_0 = t_m$. Therefore, the expectation value of $\hat{b}^*\hat{b}$ for this state at the time τ is given in the relation (2.17), but $t_0(\tau)$ should obey the inequality

$$t \leq t_0(\tau) \leq t_m . \quad (2.18)$$

Since $\langle \hat{b}^*\hat{b} \rangle_\tau$ changes in the range (2.18), for example, $\langle \hat{b}^*\hat{b} \rangle_\tau$ can change periodically in the range

$$2t - 1 \leq \langle \hat{b}^*\hat{b} \rangle_\tau \leq t_m + t - 1 . \quad (2.19)$$

Of course, it depends on the Hamiltonian. In (A), an example of the periodical change was shown. In conclusion, in order to make the idea of the phase space doubling effective in the $su(1,1)$ - and the pseudo $su(1,1)$ -algebra, the operator $(\hat{b}^*\hat{b} - \hat{a}^*\hat{a})$, i.e., \hat{T} should be a constant of motion.

§3. A new boson realization of many-fermion system obeying the $su(2)$ -algebra in terms of the $su(1,1)$ -algebra

In (A), the pseudo $su(1,1)$ -algebra in the fermion space was discussed in terms of a possible deformation of the Cooper pair obeying the $su(2)$ -algebra. In this paper, Part I, the pseudo $su(1,1)$ -algebra is treated from the side of the $su(2)$ -algebra in the frame of the boson space constructed by two kinds of bosons (\hat{a}, \hat{a}^*) and (\hat{b}, \hat{b}^*) . One of the popular boson representations of the $su(2)$ -algebra is presented by Schwinger:⁴⁾

$$\hat{S}_+ = \hat{a}^*\hat{b} , \quad \hat{S}_- = \hat{b}^*\hat{a} , \quad \hat{S}_0 = \frac{1}{2}(\hat{a}^*\hat{a} - \hat{b}^*\hat{b}) . \quad (3.1)$$

Here, $\hat{S}_{\pm,0}$ denote the generators. The Casimir operator $\hat{\mathbf{S}}^2$ is given by

$$\hat{\mathbf{S}}^2 = \hat{S}_0^2 + \frac{1}{2}(\hat{S}_-\hat{S}_+ + \hat{S}_+\hat{S}_-) = \hat{S}(\hat{S} + 1) , \quad (3.2a)$$

$$\hat{S} = \frac{1}{2}(\hat{a}^*\hat{a} + \hat{b}^*\hat{b}) , \quad [\hat{S} , \hat{S}_{\pm,0}] = 0 . \quad (3.2b)$$

We note the relation

$$\hat{S} = \hat{T}_0 - \frac{1}{2} , \quad \hat{S}_0 = -\hat{T} + \frac{1}{2} . \quad (3.3)$$

Since \hat{S} is not related to \hat{T} , the form (3.1) may be not suitable for treating the Hamiltonian (1.1) based on the phase space doubling, even if it is expressed in terms of the $su(2)$ -generators. Rather, it may be better to apply this representation to the problem of the energy transfer between the b - and the a -boson system.

The above mentioning suggests us that, in order to obtain a $su(2)$ -algebra in the boson representation for the phase space doubling, it may be necessary to connect the operator \hat{S} with \hat{T} directly. With this aim, the same idea as that in the pseudo $su(1, 1)$ -algebra (2.9) is adopted:

$$\hat{S}_-|t, t\rangle = 0, \quad \hat{S}_0|t, t\rangle = -s|t, t\rangle, \quad (3.4a)$$

$$\hat{S}_+|t, t_m\rangle = 0, \quad \hat{S}_0|t, t_m\rangle = s|t, t_m\rangle, \quad (3.4b)$$

$$\left(\hat{S}_+\right)^{t_0-t}|t\rangle = \sqrt{\frac{(2t-1)!}{(t_0+t-1)!} \cdot \frac{(t_m-t)!}{(t_m-t_0)!}} \left(\hat{T}_+\right)^{t_0-t}|t\rangle. \quad (3.4c)$$

Since the eigenvalue of \hat{S}_0 increases one by one from $-s$ to s and, then, \hat{S}_0 may be permitted to set up the relation

$$\hat{S}_0 = \hat{T}_0 + f(\hat{T}). \quad (3.5)$$

The above is the same as the case of the pseudo $su(1, 1)$ -algebra. Operating \hat{S}_0 on the states $|t, t\rangle$ and $|t, t_m\rangle$, the following relation is obtained:

$$t + f(t) = -s, \quad t_m + f(t) = s. \quad (3.6)$$

The relation (3.6) leads us to $f(t) = -(t_m + t)/2$ and $s = (t_m - t)/2$. Therefore, \hat{S}_0 can be expressed as

$$\hat{S}_0 = \hat{T}_0 - \frac{1}{2}(\hat{T}_m + \hat{T}). \quad (3.7a)$$

Later, we contact with $s = (t_m - t)/2$. If noticing the relations $\hat{T}_-|t\rangle = 0$ and $\sqrt{\hat{T}_m - \hat{T}_0}|t, t_m\rangle = 0$, the requirement (3.4) suggests the following form for \hat{S}_\pm :

$$\begin{aligned} \hat{S}_+ &= \hat{T}_+ \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \left(\sqrt{\hat{T}_0 + \hat{T} + \epsilon}\right)^{-1}, \\ \hat{S}_- &= \left(\sqrt{\hat{T}_0 + \hat{T} + \epsilon}\right)^{-1} \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \hat{T}_-. \end{aligned} \quad (3.7b)$$

In the space obeying the relation (3.4), the operators $\hat{S}_{\pm,0}$ satisfy

$$\hat{S}_-^* = \hat{S}_+, \quad \hat{S}_0^* = \hat{S}_0, \quad (3.8a)$$

$$[\hat{S}_+, \hat{S}_-] = 2\hat{S}_0, \quad [\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm. \quad (3.8b)$$

The proof of the relation (3.8b) will be shown in Part II. Certainly, $\hat{S}_{\pm,0}$ form the $su(2)$ -algebra. The Casimir operator $\hat{\mathcal{S}}^2$ can be expressed as

$$\hat{\mathcal{S}}^2 = \hat{S}(\hat{S} + 1), \quad \hat{S} = \frac{1}{2}(\hat{T}_m - \hat{T}). \quad (3.8c)$$

The operator \hat{S} is given from the form $s = (t_m - t)/2$ obtained in the relation (3.6). If $\hat{T}_m = C_m + 1 - \hat{T}$, $\hat{S}_{\pm,0}$ and \hat{S} can be expressed as follows:

$$\hat{S}_+ = \hat{a}^*\hat{b}^* \cdot \sqrt{C_m - \hat{b}^*\hat{b}} \cdot \left(\sqrt{\hat{b}^*\hat{b} + 1 + \epsilon}\right)^{-1}, \quad (3.9a)$$

$$\hat{S}_- = \left(\sqrt{\hat{b}^* \hat{b} + 1 + \epsilon} \right)^{-1} \cdot \sqrt{C_m - \hat{b}^* \hat{b}} \cdot \hat{b} \hat{a} , \quad (3.9b)$$

$$\hat{S}_0 = \frac{1}{2}(\hat{a}^* \hat{a} + \hat{b}^* \hat{b}) - \frac{1}{2}C_m , \quad (3.9c)$$

$$\hat{S} = \frac{1}{2}(\hat{a}^* \hat{a} - \hat{b}^* \hat{b}) + \frac{1}{2}C_m . \quad (3.10)$$

The above is a new boson representation of the $su(2)$ -algebra. The expressions (3.7) and (3.8c) hold in the subspace (2.8) of the space (2.5a). The detail explanation will be given in Part II. The form (3.10) suggests that the present representation may be suitable for the phase space doubling, because $\hat{b}^* \hat{b}$ and $\hat{a}^* \hat{a}$ appear in the relation of the subtraction of the b - and the a -boson number.

We have developed a new boson representation of the $su(2)$ -algebra. Such boson representations have played a role of describing various gross properties of many-fermion systems. In these studies, the investigation on the behaviors of individual fermions is of secondary importance. These have been called the boson realization of the Lie algebraic approach to many-fermion problems. The $su(2)$ -pairing model is a typical example. In this model, three quantities occupy central part of the gross properties, that is, the total number of the single-particle states $4\Omega_0$ (if follows (A), conventionally 2Ω), the total fermion number N and the seniority number ν . Therefore, in order to complete the present new boson representation in relation to the $su(2)$ -pairing model, it may be inevitable to connect t , t_0 and t_m with ν , N and Ω_0 .

First, the form $\tilde{S}_0 = (1/2)\tilde{N} - \Omega_0$ in the relation (A.3.1) is taken up in terms of the eigenvalue s_0 and N :

$$s_0 = \frac{1}{2}N - \Omega_0 . \quad (3.11a)$$

In the case $s_0 = -s$, $-s = N_{\min}/2 - \Omega_0$ and $N_{\min} = \nu$, and then, the following relation is derived:

$$s = \Omega_0 - \frac{1}{2}\nu . \quad (3.11b)$$

Further, in the case $s_0 = s$, $s = N_{\max}/2 - \Omega_0 = \Omega_0 - \nu/2$, and then, $N_{\max} = 4\Omega_0 - \nu$. Noticing the relation $[\tilde{N}, \tilde{S}_+] = 2\tilde{S}_+$, N can be given as

$$N = \nu , \nu + 2 , \dots , 4\Omega_0 - \nu (= \nu + 2(2\Omega_0 - \nu)) . \quad (3.12)$$

Since $s \geq 0$, ν is given as

$$\nu = 0 , 1 , 2 , \dots , 2\Omega_0 . \quad (3.13)$$

On the other hand, the relations (3.7a) and (3.8c) lead us to

$$s_0 = t_0 - \frac{1}{2}(t_m + t) , \quad (3.14a)$$

$$s = \frac{1}{2}(t_m - t) . \quad (3.14b)$$

Equating the relations (3.11) and (3.14) with each other, the following is obtained:

$$2t - 1 = \nu + \frac{1}{2}((2t_0 - 1) - N) , \quad (3.15a)$$

$$2t_m - 1 = (4\Omega_0 - \nu) + \frac{1}{2}((2t_0 - 1) - N) . \quad (3.15b)$$

Discussion on the relations (3.15a) and (3.15b) starts in a postulate mentioned below. The boson vacuum $|0\rangle = |t = 1/2, t_0 = 1/2\rangle$ corresponds to the fermion vacuum $|0\rangle = |\nu = 0, N = 0\rangle$. This postulate suggests that the relations (2.8) and (2.13) give us

$$2t - 1 = \nu . \quad (3.16a)$$

The relations (3.15a) and (3.15b) lead us to

$$2t_0 - 1 = N , \quad (3.16b)$$

i.e.,

$$2t_m - 1 = 4\Omega_0 - \nu . \quad (3.16c)$$

Since $2t_m - 1 = 4\Omega_0 - (2t - 1)$, the following relation is obtained:

$$t_m = (2\Omega_0 + 1) - t . \quad (3.17)$$

The relations (3.14b) and (3.17) give

$$s = \left(\Omega_0 + \frac{1}{2} \right) - t . \quad (3.18a)$$

The maximum values of t and ν are μ and $2\Omega_0$, respectively, and the relations (3.13) and (3.15a) give us

$$2\mu - 1 = 2\Omega_0 , \quad \text{i.e.,} \quad \mu = \Omega_0 + \frac{1}{2} . \quad (3.18b)$$

The above argument presents us the operator form of the $su(2)$ -algebra. First, the seniority and the total fermion number operator $\hat{\nu}$ and \hat{N} , respectively, are introduced in the form

$$\hat{\nu} = 2\hat{T} - 1 = -\hat{a}^*\hat{a} + \hat{b}^*\hat{b} , \quad (3.19a)$$

$$\hat{N} = 2\hat{T}_0 - 1 = \hat{a}^*\hat{a} + \hat{b}^*\hat{b} . \quad (3.19b)$$

Further, \hat{T}_m is given as

$$\hat{T}_m = (2\Omega_0 + 1) - \hat{T} = 2\Omega_0 + \frac{1}{2} + \frac{1}{2}(\hat{a}^*\hat{a} - \hat{b}^*\hat{b}) . \quad (3.20)$$

The operators $\hat{\mathcal{S}}_{\pm,0}$ and $\hat{\mathcal{S}}$ can be expressed as

$$\hat{\mathcal{S}}_+ = \hat{T}_+ \cdot \sqrt{(2\Omega_0 + 1) - (\hat{T}_0 + \hat{T})} \cdot \left(\sqrt{\hat{T}_0 + \hat{T} + \epsilon} \right)^{-1}$$

$$= \hat{a}^* \hat{b}^* \cdot \sqrt{2\Omega_0 - \hat{b}^* \hat{b}} \cdot \left(\sqrt{\hat{b}^* \hat{b} + 1 + \epsilon} \right)^{-1}, \quad (3.21a)$$

$$\begin{aligned} \hat{\mathcal{S}}_- &= \left(\sqrt{\hat{T}_0 + \hat{T} + \epsilon} \right)^{-1} \cdot \sqrt{(2\Omega_0 + 1) - (\hat{T}_0 + \hat{T})} \cdot \hat{T}_- \\ &= \left(\sqrt{\hat{b}^* \hat{b} + 1 + \epsilon} \right)^{-1} \cdot \sqrt{2\Omega_0 - \hat{b}^* \hat{b}} \cdot \hat{b} \hat{a}, \end{aligned} \quad (3.21b)$$

$$\hat{\mathcal{S}}_0 = \hat{T}_0 - \left(\Omega_0 + \frac{1}{2} \right) = \frac{1}{2}(\hat{a}^* \hat{a} + \hat{b}^* \hat{b}) - \Omega_0, \quad (3.21c)$$

$$\hat{\mathcal{S}} = \left(\Omega_0 + \frac{1}{2} \right) - \hat{T} = \frac{1}{2}(\hat{a}^* \hat{a} - \hat{b}^* \hat{b}) + \Omega_0. \quad (3.22)$$

The above corresponds to the case $C_m = 2\Omega_0$ in the relation (3.7) and (3.9). Needless to say, the expressions (3.21) and (3.22) hold in the subspace (2.8) of the space (2.5a). The detail will be mentioned in Part II. The expression (3.21) and (3.22) form the third boson representation of the $su(2)$ -algebra. Of course, the first and the second are the Holstein-Primakoff and the Schwinger boson representation, respectively.^{3),4)}

§4. Various deformations of the $su(2)$ -algebra in the third boson representation

In this section, we will investigate various deformations of the $su(2)$ -algebra developed in §3. For this aim, let us rewrite the generators $\hat{\mathcal{S}}_{\pm}$ shown in the relation (3.7b) as follows:

$$\hat{\mathcal{S}}_+ = \hat{T}_+ \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \sqrt{1 + (\hat{T}_0 - \hat{T})} \cdot \left(\sqrt{\hat{T}_- \hat{T}_+ + \epsilon} \right)^{-1}, \quad (4.1a)$$

$$\hat{\mathcal{S}}_- = \left(\sqrt{\hat{T}_- \hat{T}_+ + \epsilon} \right)^{-1} \cdot \sqrt{1 + (\hat{T}_0 - \hat{T})} \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \hat{T}_-. \quad (4.1b)$$

The operator $\hat{\mathcal{S}}_0$ is unchanged from the form (3.7a) and $\hat{T}_- \hat{T}_+$ is given as

$$\hat{T}_- \hat{T}_+ = (\hat{T}_0 + \hat{T}) (\hat{T}_0 - \hat{T} + 1) = (\hat{T}_0 + \hat{T}) \left(1 + (\hat{T}_0 - \hat{T}) \right). \quad (4.2a)$$

If \hat{T}_0 is replaced with $(\hat{T}_0 - 1)$, $\hat{T}_- \hat{T}_+$ becomes to $\hat{T}_+ \hat{T}_-$:

$$\hat{T}_+ \hat{T}_- = (\hat{T}_0 - \hat{T}) (\hat{T}_0 + \hat{T} - 1). \quad (4.2b)$$

With the use of the relation (4.2a), it may be easily verified that the form (4.1) is equivalent to the relation (3.7b).

The discussion starts in the introduction of the operator $\hat{\mathcal{R}}_{\pm}$ deformed from $\hat{\mathcal{S}}_{\pm}$:

$$\hat{\mathcal{R}}_+ = \hat{T}_+ \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \sqrt{\hat{Q}_p + p(\hat{T}_0 - \hat{T})} \cdot \left(\sqrt{\hat{T}_- \hat{T}_+ + \epsilon} \right)^{-1}, \quad (4.3a)$$

$$\hat{\mathcal{R}}_- = \left(\sqrt{\hat{T}_- \hat{T}_+ + \epsilon} \right)^{-1} \cdot \sqrt{\hat{Q}_p + p(\hat{T}_0 - \hat{T})} \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \hat{T}_-. \quad (4.3b)$$

Here, \hat{Q}_p is a function of \hat{T} :

$$\hat{Q}_p = Q_p(\hat{T}) , \quad \hat{Q}_p|t, t_0\rangle = Q_p(t)|t, t_0\rangle , \quad Q_p(t) = q_{p,t} . \quad (4.4)$$

From the outside, we must fix the concrete form of \hat{Q}_p and corresponding to the form of \hat{Q}_p , the deformation is determined. Without loss of generality, the following conditions are added:

$$p = \pm 1, 0 \quad \text{and if } p = 0 , \quad \hat{Q}_0 = 1 , \quad \text{i.e., } q_{0,t} = 1 . \quad (4.5)$$

The operator $(\hat{Q}_p + p(\hat{T}_0 - \hat{T}))$ should be positive-definite and it can be shown in the form

$$\sqrt{\hat{Q}_p + p(\hat{T}_0 - \hat{T})}|t, t\rangle = \sqrt{q_{p,t}}|t, t\rangle , \quad (4.6a)$$

$$\sqrt{\hat{Q}_p + p(\hat{T}_0 - \hat{T})}|t, t_m\rangle = \sqrt{q_{p,t} + p(t_m - t)}|t, t_m\rangle . \quad (4.6b)$$

The relation (4.6) leads us to

$$q_{p,t} \geq 0 , \quad (4.7a)$$

$$q_{p,t} + p(t_m - t) \geq 0 , \quad \text{i.e., } q_{1,t} \geq -(t_m - t) , \quad q_{-1,t} \geq t_m - t . \quad (4.7b)$$

If $q_{p,t} = 0$, $\sqrt{\hat{Q}_p + p(\hat{T}_0 - \hat{T})}|t, t\rangle = 0$ and, then, $\hat{\mathcal{R}}_+|t, t\rangle = 0$, which is not interesting, because of $\hat{\mathcal{R}}_-|t, t\rangle = 0$. Therefore, in the relation (4.7a), $q_{p,t} \geq 0$ should be changed to $q_{p,t} > 0$. From the above argument, $q_{p,t}$ should obey the condition for the positive-definiteness

$$q_{1,t} > 0 , \quad q_{-1,t} \geq t_m - t . \quad (4.8)$$

The operators $\hat{\mathcal{R}}_{\pm}$ are reduced to $\hat{\mathcal{S}}_{\pm}$ shown in the relation (4.1):

$$\text{If } \hat{Q}_1 = 1 , \quad \hat{\mathcal{R}}_{\pm} = \hat{\mathcal{S}}_{\pm} . \quad (4.9)$$

The products of $\hat{\mathcal{R}}_+$ and $\hat{\mathcal{R}}_-$ are expressed as

$$\hat{\mathcal{R}}_+ \hat{\mathcal{R}}_- = (\hat{T}_m - \hat{T}_0 + 1) (\hat{Q}_p + p(\hat{T}_0 - \hat{T} - 1)) \left(1 - \frac{\epsilon}{\hat{T}_+ \hat{T}_- + \epsilon}\right) , \quad (4.10a)$$

$$\hat{\mathcal{R}}_- \hat{\mathcal{R}}_+ = (\hat{T}_m - \hat{T}_0) (\hat{Q}_p + p(\hat{T}_0 - \hat{T})) \left(1 - \frac{\epsilon}{\hat{T}_- \hat{T}_+ + \epsilon}\right) . \quad (4.10b)$$

In the case $p = \pm 1$, the relation (4.10) gives us the commutation relation

$$[\hat{\mathcal{R}}_+ , \hat{\mathcal{R}}_-] = p \cdot 2\hat{\mathcal{R}}_0 - (\hat{T}_m - \hat{T} + 1) \sum_t |t\rangle \langle t| . \quad (4.11)$$

Here, $\hat{\mathcal{R}}_0$ is defined as

$$\hat{\mathcal{R}}_0 = \hat{T}_0 - \frac{1}{2} (\hat{T}_m + \hat{T} - p\hat{Q}_p + 1) , \quad [\hat{\mathcal{R}}_0 , \hat{\mathcal{R}}_{\pm}] = \pm \hat{\mathcal{R}}_{\pm} . \quad (4.12)$$

Under the condition (4.9), $\hat{\mathcal{R}}_0$ is reduced to $\hat{\mathcal{S}}_0$:

$$\text{If } \hat{Q}_1 = 1, \quad \hat{\mathcal{R}}_0 = \hat{\mathcal{S}}_0. \quad (4.13)$$

The operator $\hat{\mathcal{R}}^2$ corresponding to the Casimir operator $\hat{\mathcal{S}}^2$ is expressed in the form

$$\begin{aligned} \hat{\mathcal{R}}^2 &= \hat{\mathcal{R}}_0^2 + p \cdot \frac{1}{2} \left(\hat{\mathcal{R}}_- \hat{\mathcal{R}}_+ + \hat{\mathcal{R}}_+ \hat{\mathcal{R}}_- \right) \\ &= F_p(\hat{\mathcal{R}}) + \frac{1}{2} \left(\hat{T}_m - \hat{T} + 1 \right) \left(\hat{Q}_p - p \right) \sum_t |t\rangle \langle t|. \end{aligned} \quad (4.14)$$

Here, $\hat{\mathcal{R}}$ and $F_p(\hat{\mathcal{R}})$ are given as

$$(i) \quad \hat{\mathcal{R}} = \pm \frac{1}{2} \left(\hat{T}_m - \hat{T} + p\hat{Q}_p \mp 1 \right), \quad F_p(\hat{\mathcal{R}}) = \hat{\mathcal{R}} \left(\hat{\mathcal{R}} + 1 \right), \quad (4.15a)$$

$$(ii) \quad \hat{\mathcal{R}} = \pm \frac{1}{2} \left(\hat{T}_m - \hat{T} + p\hat{Q}_p \pm 1 \right), \quad F_p(\hat{\mathcal{R}}) = \hat{\mathcal{R}} \left(\hat{\mathcal{R}} - 1 \right), \quad (4.15b)$$

Under the condition (4.9), the upper sign of $\hat{\mathcal{R}}$ in (i) is reduced to $\hat{\mathcal{S}}$:

$$\text{If } \hat{Q}_1 = 1, \quad \hat{\mathcal{R}} = \hat{\mathcal{S}} \quad \text{and} \quad \hat{\mathcal{R}}^2 = F_1(\hat{\mathcal{R}}) = \hat{\mathcal{S}}^2 = \hat{\mathcal{S}} \left(\hat{\mathcal{S}} + 1 \right). \quad (4.16)$$

The above argument suggests that $\hat{\mathcal{R}}$ shown in the relation (4.15) can be regarded as the operator which plays the same role as that of $\hat{\mathcal{S}}$. Then, it may be permitted to require the condition

$$r_t \geq 0. \quad \left(\hat{\mathcal{R}}|t, t_0\rangle = r_t|t, t_0\rangle \right) \quad (4.17)$$

The above is formal aspect of the case $p = \pm 1$. Later, we will discuss some concrete examples.

Next, we discuss the case $p = 0$. Under the condition (4.5), the relation (4.10) is reduced to

$$\hat{\mathcal{R}}_+ \hat{\mathcal{R}}_- = \hat{T}_m - \hat{T} + 1 - \left(\hat{T}_m - \hat{T} + 1 \right) \sum_t |t\rangle \langle t|, \quad (4.18a)$$

$$\hat{\mathcal{R}}_- \hat{\mathcal{R}}_+ = \hat{T}_m - \hat{T}. \quad (4.18b)$$

In this case, $\hat{\mathcal{R}}_0$ cannot be defined, because of the commutation relation

$$[\hat{\mathcal{R}}_+, \hat{\mathcal{R}}_-] = 1 - \left(\hat{T}_m - \hat{T} + 1 \right) \sum_t |t\rangle \langle t|. \quad (4.19)$$

But, it should be noticed that \hat{T}_0 plays the same role as that of $\hat{\mathcal{R}}_0$ in the case $p = \pm 1$:

$$[\hat{T}_0, \hat{\mathcal{R}}_{\pm}] = \pm \hat{\mathcal{R}}_{\pm}. \quad (4.20)$$

Of course, $\hat{\mathcal{R}}^2$ cannot be defined. The operators $\hat{\mathcal{R}}_{\pm}$ can be expressed as

$$\hat{\mathcal{R}}_+ = \hat{T}_+ \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \left(\sqrt{\hat{T}_- \hat{T}_+ + \epsilon} \right)^{-1}, \quad (4.21a)$$

$$\hat{\mathcal{R}}_- = \left(\sqrt{\hat{T}_- \hat{T}_+ + \epsilon} \right)^{-1} \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \hat{T}_-. \quad (4.21b)$$

Although $\hat{\mathcal{R}}_{\pm}$ do not form any algebra, $\hat{\mathcal{R}}_+$ plays a role of the raising operator for constructing the orthogonal set. It is easily seen that there exist the relations $\hat{\mathcal{R}}_-|t, t\rangle = 0$ and $\hat{\mathcal{R}}_+|t, t_m\rangle = 0$ and the state $|t, t_0\rangle$ is of the form $(\hat{\mathcal{R}}_+)^{t_0-t}|t\rangle$.

As was already promised, we show concrete examples for the case $p = \pm 1$ and examine the following cases:

$$(i) \quad \hat{\mathcal{R}}_0|t, t\rangle = -\hat{\mathcal{R}}|t, t\rangle (= -r_t|t, t\rangle), \quad (4.22a)$$

$$(ii) \quad \hat{\mathcal{R}}_0|t, t_m\rangle = \hat{\mathcal{R}}|t, t_m\rangle (= r_t|t, t_m\rangle), \quad (4.22b)$$

$$(iii) \quad \hat{\mathcal{R}}_0|t, t\rangle = \hat{\mathcal{R}}|t, t\rangle (= r_t|t, t\rangle). \quad (4.22c)$$

These three may be regarded as the cases in which traces of the original $su(2)$ - and $su(1, 1)$ -algebras are left. After rather lengthy consideration, the following results are obtained: In the case (i), $\hat{Q}_1 = 1$, i.e., $q_{1,t} = 1$, which leads us to

$$\hat{\mathcal{R}} = \frac{1}{2} (\hat{T}_m - \hat{T}), \quad \hat{\mathcal{R}}_0 = \hat{T}_0 - \frac{1}{2} (\hat{T}_m + \hat{T}), \quad \hat{\mathcal{R}}^2 = \hat{\mathcal{R}} (\hat{\mathcal{R}} + 1) \quad (4.23a)$$

In the case (ii), $q_{1,t} > 0$, which gives

$$\begin{aligned} \hat{\mathcal{R}} &= \frac{1}{2} (\hat{T}_m - \hat{T} + \hat{Q}_1 - 1), \quad \hat{\mathcal{R}}_0 = \hat{T}_0 - \frac{1}{2} (\hat{T}_m + \hat{T} - \hat{Q}_1 + 1), \\ \hat{\mathcal{R}}^2 &= \hat{\mathcal{R}} (\hat{\mathcal{R}} + 1) - \frac{1}{2} (\hat{T}_m - \hat{T} + 1) (\hat{Q}_1 - 1) \sum_t |t\rangle \langle t|. \end{aligned} \quad (4.23b)$$

In the case (iii), it is impossible to find any case which satisfies the conditions (4.8) and (4.17).

It is important to see that the case (i) is included in the case (ii). If $\hat{Q}_1 = 1$ in the case (ii), it is nothing but the case (i) and this case corresponds to the $su(2)$ -algebra already discussed. If $\hat{Q}_1 \neq 1$, there does not exist the relation $\hat{\mathcal{R}}_0|t, t\rangle = -\hat{\mathcal{R}}|t, t\rangle$. For example, if $\hat{Q}_1 = 2\hat{T}$, the operator $\hat{Q}_p + p(\hat{T}_0 - \hat{T})$ for $p = 1$ becomes $(\hat{T} + \hat{T}_0)$. Then, in this case, $\hat{\mathcal{R}}_{\pm,0}$ can be expressed in the form

$$\hat{\mathcal{R}}_+ = \hat{T}_+ \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \left(\sqrt{\hat{T}_0 - \hat{T} + 1 + \epsilon} \right)^{-1}, \quad (4.24a)$$

$$\hat{\mathcal{R}}_- = \left(\sqrt{\hat{T}_0 - \hat{T} + 1 + \epsilon} \right)^{-1} \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \hat{T}_-, \quad (4.24b)$$

$$\hat{\mathcal{R}}_0 = \hat{T}_0 - \frac{1}{2} (\hat{T}_m - \hat{T} + 1). \quad (4.24c)$$

$$\hat{\mathcal{R}} = \frac{1}{2} (\hat{T}_m - \hat{T} - 1), \quad \hat{\mathcal{R}}^2 = \hat{\mathcal{R}} (\hat{\mathcal{R}} + 1) - \frac{1}{2} (\hat{T}_m - \hat{T} + 1) (2\hat{T} - 1) \sum_t |t\rangle\langle t|. \quad (4.25)$$

Of course, the following relations are obtained:

$$[\hat{\mathcal{R}}_+, \hat{\mathcal{R}}_-] = 2\hat{\mathcal{R}}_0 - (\hat{T}_m - \hat{T} + 1) (2\hat{T} - 1) \sum_t |t\rangle\langle t|, \quad (4.26)$$

$$\hat{\mathcal{R}}_0 |t, t\rangle = -\frac{1}{2} (t_m - 3t + 1) |t, t\rangle \neq -\hat{\mathcal{R}} |t, t\rangle. \quad (\text{if } t > 1/2) \quad (4.27)$$

The case $t = 1/2$ is reduced to the $su(2)$ -algebra $(\hat{\mathcal{S}}_{\pm,0})$. The above argument may be permitted to call $\hat{Q}_1 \neq 1$ as a pseudo $su(2)$ -algebra.

The $(t_0 - t)$ -time operation of $\hat{\mathcal{R}}_+$ on the state $|t\rangle$ is given in the form

$$(\hat{\mathcal{R}}_+)^{t_0-t} |t\rangle = \sqrt{\frac{(t_m - t)!}{(t_m - t_0)!}} \sqrt{\prod_{k=0}^{t_m-t-1} (q_{p,t} + pk)} \sqrt{\frac{(2t-1)!}{(t_0 - t)!(t_0 + t - 1)!}} (\hat{T}_+)^{t_0-t} |t\rangle. \quad (4.28)$$

Here, $\prod_{k=0}^{t_m-t-1} (q_{p,t} + pk)$ can be expressed in terms of the gamma-function:

$$\prod_{k=0}^{t_m-t-1} (q_{p,t} + pk) = p^n \frac{\Gamma(q_{p,t}/p + n)}{\Gamma(q_{p,t}/p)}. \quad (n = t_0 - t) \quad (4.29)$$

If we intend to describe the system under investigation exactly, it may be enough to use the orthogonal state $(\hat{T}_+)^n |t\rangle$ ($n = 0, 1, 2, \dots, t_m - t$). However, if we adopt approximation such as in (A), the above idea of the deformation becomes useful. We will consider the case on the following state:

$$|\phi_{p,t}\rangle = \frac{1}{\sqrt{\Gamma_{p,t}}} \exp(z\hat{\mathcal{R}}_+) |t\rangle. \quad (\langle\phi_{p,t}|\phi_{p,t}\rangle = 1) \quad (4.30)$$

Here, z and $\Gamma_{p,t}$ denote a complex parameter and the normalization constant, respectively. In such problem, it is indispensable which form is chosen for $\hat{\mathcal{R}}_+$. In (A), as $\hat{\mathcal{R}}_+$, \hat{T}_+ , i.e., \hat{T}_+ itself was used. The norm of the state $(\hat{\mathcal{R}}_+)^n |t\rangle$ is given by

$$\begin{aligned} \langle t | (\hat{\mathcal{R}}_-)^n \cdot (\hat{\mathcal{R}}_+)^n |t\rangle &= \frac{(t_m - t)!}{(t_m - t - n)!} \prod_{k=0}^{n-1} (q_{p,t} + pk) \cdot \frac{(2t-1)!}{n!(2t-1+n)!} \langle t | (\hat{T}_-)^n \cdot (\hat{T}_+)^n |t\rangle \\ &= \frac{(t_m - t)!}{(t_m - t - n)!} \prod_{k=0}^{n-1} (q_{p,t} + pk), \end{aligned} \quad (4.31)$$

$$\langle t | (\hat{T}_-)^n \cdot (\hat{T}_+)^n |t\rangle = \frac{(2t-1+n)!n!}{(2t-1)!}. \quad (4.32)$$

Then, $\Gamma_{p,t}$ is expressed as a function of x in the following:

$$\Gamma_{p,t} = \sum_{n=0}^{t_m-t} (-)^n \binom{t_m-t}{n} \frac{\Gamma(q_{p,t}/p + n)}{n! \Gamma(q_{p,t}/p)} (-px)^n$$

$$= G_{t_m-t}(q_{p,t}/p - (t_m - t), 1; -px) , \quad (4.33)$$

$$x = |z|^2 . \quad (4.34)$$

Here, the relation (4.29) was used. The function G_{t_m-t} is Jacobi polynomial. At the limit $p \rightarrow 0$ and $q_{p,t} \rightarrow 1$ for the expression (4.33) is reduced to

$$\Gamma_{0,t} = \lim_{\substack{p \rightarrow 0 \\ q_{p,t} \rightarrow 1}} \Gamma_{p,t} = \sum_{n=0}^{t_m-t} \frac{(-)^n}{n!} \binom{t_m-t}{n} (-x)^n = L_{t_m-t}(-x) . \quad (4.35)$$

The function L_{t_m-t} denotes Laguerre polynomial. Here, the following relation was used:

$$\lim_{\substack{p \rightarrow 0 \\ q_{p,t} \rightarrow 1}} \frac{\Gamma(q_{p,t}/p + n)}{\Gamma(q_{p,t}/p)} (-px)^n = (-x)^n . \quad (4.36)$$

With the use of the relation (4.29), we can obtain the expression (4.35) directly.

The expression (4.33) is too complicated to apply it to any concrete problem. This indicates that idea for the approximation must be searched. For this aim, three points for $\Gamma_{p,t}$ must be pointed out. First is the case $q_{1,t} = 1$, which leads us to the simple form:

$$\Gamma_{1,t} = (1+x)^{t_m-t} . \quad (4.37)$$

Second is the maximum power of the polynomial $\Gamma_{p,t}$ for x :

$$\text{the maximum power} = t_m - t . \quad (4.38)$$

It is independent of the choice of $q_{p,t}$. Third is related to the behavior of $\Gamma_{p,t}$ near $x = 0$. In the region $x \sim 0$, $\Gamma_{p,t}$ can be expressed as

$$\Gamma_{p,t} = 1 + \Gamma_{p,t}^{(1)} x + \frac{1}{2} \Gamma_{p,t}^{(2)} x^2 + \cdots , \quad (4.39a)$$

$$\Gamma_{p,t}^{(1)} = (t_m - t) q_{p,t} , \quad \Gamma_{p,t}^{(2)} = \frac{1}{2} (t_m - t)(t_m - t - 1) q_{p,t} (q_{p,t} + p) . \quad (4.39b)$$

Concerning $\Gamma_{p,t}(x)$, the above-mentioned three points suggest us the following approximation:

$$\begin{aligned} \Gamma^a(x) &= (1 + Cx)^k (1 + Dx)^l , \\ k, l &: \text{positive integers} , \quad k + l = m (= t_m - t) . \end{aligned} \quad (4.40)$$

In order to take into account the difference between C and D , $\Gamma^a(x)$ should be treated in the region

$$1 \leq l \leq m - 1 . \quad (4.41)$$

Hereafter, in order to avoid unnecessary complication, we will omit the index (p, t) and use the symbol $m (= t_m - t)$. We determine C and D so as to make the coefficients of the terms x and x^2 in the form (4.40) agree with those in the relation (4.39):

$$C = q \left(1 \mp \sqrt{\frac{l}{k} \cdot \frac{m-1}{2} \cdot \zeta} \right) , \quad D = q \left(1 \pm \sqrt{\frac{k}{l} \cdot \frac{m-1}{2} \cdot \zeta} \right) , \quad (4.42)$$

$$\zeta = 1 - \frac{p}{q} \ (\geq 0) . \quad (4.43a)$$

The condition $\zeta \geq 0$ and the relations (4.5) and (4.8) give us the inequality for ζ :

$$\begin{aligned} 0 \leq \zeta \leq 1 + \frac{1}{m} , \\ \text{i.e.,} \\ 0 \leq \zeta < 1 \ (p = 1) , \quad \zeta = 1 \ (p = 0) , \quad 1 < \zeta \leq 1 + \frac{1}{m} \ (p = -1) . \end{aligned} \quad (4.43b)$$

The function $\Gamma(x)(= \Gamma_{p,t}(x))$ is a polynomial, in which the coefficient of each term is positive and, then, $\Gamma(x) = 0$ does not have any root in the range $x \geq 0$, i.e., $\Gamma(x) > 0 \ (x \geq 0)$. Therefore, we require the condition $\Gamma^a(x) > 0 \ (x \geq 0)$, that is,

$$C > 0 , \quad D > 0 . \quad (4.44)$$

The quantities C and D are symmetric with each other for k, l, \mp and \pm in the relation (4.42) and for D , we can adopt the form with the upper in the sign \pm :

$$D = q \left(1 + \sqrt{\frac{k}{l} \cdot \frac{m-1}{2} \cdot \zeta} \right) \ (> 0) . \quad (4.45a)$$

Therefore, automatically, C becomes of the form

$$C = q \left(1 - \sqrt{\frac{l}{k} \cdot \frac{m-1}{2} \cdot \zeta} \right) \ (> 0) . \quad (4.45b)$$

The expression (4.45a) satisfies $D > 0$. The condition $C > 0$ is reduced to

$$l < \lambda_0 , \quad \lambda_0 = \frac{m}{1 + \frac{m-1}{2} \cdot \zeta} . \quad (4.46)$$

It should be noted that l is a positive integer, but, λ_0 is generally not an integer. We introduce l_0 which is the nearest integer to λ_0 under the condition $l_0 < \lambda_0$. Then, the following inequality is obtained:

$$0 < \lambda_0 - l_0 \leq 1 , \quad l = l_0, l_0 - 1, \dots, 2, 1. \quad (4.47)$$

With the use of the relation (4.46), the inequality (4.47) can be rewritten in the form

$$\begin{aligned} \zeta_{\min} \leq \zeta \leq \zeta_{\max} , \\ \zeta_{\min} = \frac{2(m-1-l_0)}{(m-1)(l_0+1)} , \quad \zeta_{\max} = \frac{2(m-l_0)}{(m-1)l_0} . \end{aligned} \quad (4.48)$$

Some concrete cases of the inequality are summarized in Table I. We can see that the case ($p = 1, q = 1$) reduces to $\zeta = 0$ and $C = D = 1$. In this case, the relation (4.40) becomes to the form (4.37) independent of the choice of l . Of course, this

Table I. Parameter set

l_0	ζ_{\min}	ζ_{\max}	l
$m-1$	0	$\frac{2}{(m-1)^2}$	$m-1, m-2, \dots, 2, 1$
$m-2$	$\frac{2}{(m-1)^2}$	$\frac{4}{(m-1)(m-2)}$	$m-2, m-3, \dots, 2, 1$
\vdots	\vdots	\vdots	\vdots
$\frac{m+1}{2}$	$\frac{2(m-3)}{(m-1)(m+3)}$	$\frac{2}{m+1}$	$\frac{m+1}{2}, \frac{m-1}{2}, \dots, 2, 1$ (m : odd)
$\frac{m}{2}$	$\frac{2(m-2)}{(m-1)(m+2)}$	$\frac{2}{m-1}$	$\frac{m}{2}, \frac{m-2}{2}, \dots, 2, 1$ (m : even)
$\frac{m-1}{2}$	$\frac{2}{m+1}$	$\frac{2(m+1)}{(m-1)^2}$	$\frac{m-1}{2}, \frac{m-3}{2}, \dots, 2, 1$ (m : odd)
\vdots	\vdots	\vdots	\vdots
2	$\frac{2(m-3)}{3(m-1)}$	$\frac{m-2}{m-1}$	2, 1
1	$\frac{m-2}{m-1}$	$1 + \frac{1}{m}$	1

case corresponds to the $su(2)$ -algebra. The result in any other case depends on the choice of l . Up to the present, we have no idea how to determine the value of l . In relation to a certain approximation adopted in next section, the case $l = 1$ may be the most reasonable.

The expectation values of \hat{T}_0 , $\hat{\mathcal{R}}_+$ and $\hat{\mathcal{R}}_-$ for $|\phi\rangle$ given in the relation (4.30) are calculated in the form

$$\langle \phi | \hat{T}_0 | \phi \rangle = t + \Lambda(x) , \quad (4.49a)$$

$$\langle \phi | \hat{\mathcal{R}}_+ | \phi \rangle = z^* \frac{\Lambda(x)}{x} , \quad \langle \phi | \hat{\mathcal{R}}_- | \phi \rangle = z \frac{\Lambda(x)}{x} . \quad (4.49b)$$

Here, $\Lambda(x)$ is defined as

$$\Lambda(x) = \frac{x \frac{d\Gamma(x)}{dx}}{\Gamma(x)} . \quad (4.50)$$

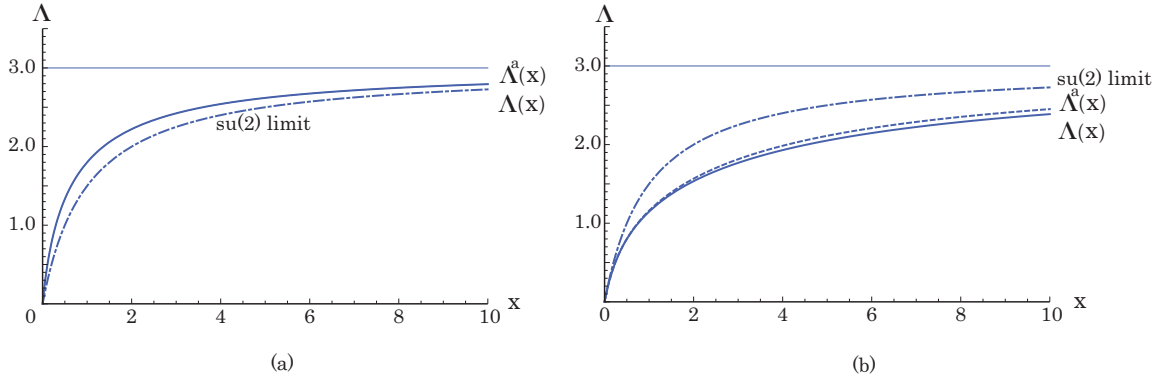


Fig. 1. Figure (a) shows the comparison of $\Lambda^a(x)$ with $\Lambda(x)$ in Eq. (4.52) for the case $k = m-1$, $l = 1$ with $m = 3$. Here, $p = 1$ and $q = 2$ are adopted. In this case, $\Lambda^a(x)$ and $\Lambda(x)$ almost overlap one another. Figure (b) shows the comparison of $\Lambda^a(x)$ (dotted curve) with $\Lambda(x)$ (solid curve) in Eq. (4.53) with $p = 0$ and $q = 1$. On both the figures, the dash-dotted curves represent the $su(2)$ limit.

In the case of the approximate form of $\Lambda(x)$, $\Lambda^a(x)$ is given in the form

$$\Lambda^a(x) = \frac{kCx}{1+Cx} + \frac{lDx}{1+Dx} = m \left[1 - \left(\frac{k/m}{1+Cx} + \frac{l/m}{1+Dx} \right) \right]. \quad (4.51)$$

The relation (4.51) will play a central role in next section. The relation (4.34) and (4.35) lead us to the following expression for $\Lambda(x)$:

$$\Lambda(x) = m \left(1 - \frac{G_{m-1}(q/p - m + 1, 1, -px)}{G_m(q/p - m, 1, -px)} \right), \quad (4.52)$$

$$\Lambda(x) = m \left(1 - \frac{L_{m-1}(-x)}{L_m(-x)} \right). \quad (4.53)$$

Figure 1 shows the comparison of $\Lambda^a(x)$ for the case ($k = m - 1$, $l = 1$) and $\Lambda(x)$.

§5. Application

As was mentioned in §1, the aim of this paper is to formulate a new boson representation of the $su(2)$ -algebra and its deformation, in which the idea of the phase space doubling is applied straightforwardly. In this section, in order to demonstrate our idea, we will apply the present form to the case of a simple boson model. This model is essentially the same as that discussed in §7 in (A). We pay an attention to the boson Hamiltonian

$$\hat{H}_b = \omega \hat{b}^* \hat{b}. \quad (5.1)$$

The Hamiltonian \hat{H}_b is nothing but \hat{H}_{intr} introduced in the relation (1.1). Following the idea of the phase space doubling, we introduce another boson Hamiltonian $\hat{H}_a = \omega \hat{a}^* \hat{a}$ and set up the form

$$\hat{H}_{ba} = \hat{H}_b - \hat{H}_a = \omega (\hat{b}^* \hat{b} - \hat{a}^* \hat{a}). \quad (5.2)$$

Of course, \hat{H}_a plays a role of \hat{H}_{extr} and $\hat{H}_{ba} = \hat{H}_0$. As for the interaction between two boson systems, \hat{V}_{ba} , we adopt the following form:

$$\hat{V}_{ba} = -i\gamma (\hat{a}^* \hat{b}^* \cdot f(\hat{a}^* \hat{a}, \hat{b}^* \hat{b}) - f(\hat{a}^* \hat{a}, \hat{b}^* \hat{b}) \cdot \hat{b} \hat{a}). \quad (5.3)$$

Here, γ denotes the interaction strength. For example, the case $f(\hat{a}^* \hat{a}, \hat{b}^* \hat{b}) = 1$ corresponds to the $su(1,1)$ -algebraic model investigated in Refs.5), 6), 7). As for $\hat{a}^* \hat{b}^* \cdot f(\hat{a}^* \hat{a}, \hat{b}^* \hat{b})$, we adopt the operator $\hat{\mathcal{R}}_+$ shown in the relation (4.3) and, then, the Hamiltonian \hat{H} is given by

$$\hat{H} = \hat{H}_{ba} + \hat{V}_{ba} = \omega (2\hat{T} - 1) - i\gamma (\hat{\mathcal{R}}_+ - \hat{\mathcal{R}}_-). \quad (5.4)$$

It should be noted that \hat{H} does not mean the total energy. It may be clear that \hat{T} is a constant of motion. The above is our model discussed in this paper.

For treating the Hamiltonian (5.4), we follow the same method as that in (A). Regarding $|\phi\rangle$ as a time-dependent variational state, we set up the following variational equation:

$$\delta \int \langle \phi | i\partial_\tau - \hat{H} | \phi \rangle d\tau = 0 . \quad (5.5)$$

Here, in order to avoid confusion between the time variable and the quantum number t , we will use τ for the time variable. If z and z^* are regarded as time-dependent variational parameters, the variational equation (5.5) leads us to the following equation:

$$\dot{z} = -\gamma \left[1 - \frac{z^2}{x} \left(1 - \frac{\Lambda(x)}{x \frac{d\Lambda(x)}{dx}} \right) \right] , \quad \dot{z}^* = -\gamma \left[1 - \frac{z^{*2}}{x} \left(1 - \frac{\Lambda(x)}{x \frac{d\Lambda(x)}{dx}} \right) \right] . \quad (5.6)$$

The expectation value of \hat{H} , \mathcal{H} , is given in the form

$$\begin{aligned} \mathcal{H} &= \langle \phi | \hat{H} | \phi \rangle = \omega(2t - 1) - i\gamma(\mathcal{R}_+ - \mathcal{R}_-) \\ &= \omega(2t - 1) - \gamma i(z^* - z) \frac{\Lambda(x)}{x} . \end{aligned} \quad (5.7)$$

Here, \mathcal{R}_\pm denote the expectation values of $\hat{\mathcal{R}}_\pm$. The detail can be found in (A).

The present system is of two dimension and, therefore, there exist two constants of motion. One is the quantum number t and the second, which will be denoted as κ , is given through the relation

$$i(z^* - z) \frac{\Lambda(x)}{x} = 2\kappa . \quad (5.8)$$

It may be self-evident, because \mathcal{H} itself shown in the relation (5.8) is a constant of motion. If z is expressed in the form $z = u + iv$, we have

$$i(z^* - z) = 2v . \quad (5.9)$$

In (A), we learned that, instead of x , it may be convenient to adopt the variable y defined as

$$y = \frac{\Lambda(x)}{x} . \quad (x = |z|^2 = u^2 + v^2) \quad (5.10)$$

Inversely solving, x can be expressed as a function of y . Then, v can be expressed in the form

$$v = \frac{\kappa}{y} , \quad \text{i.e.,} \quad yu = \pm \sqrt{xy^2 - \kappa^2} . \quad (5.11)$$

With the use of the relation (5.6), \dot{x} can be given as

$$\dot{x} = -2\gamma \frac{\Lambda(x)}{x \frac{d\Lambda(x)}{dx}} \cdot u . \quad (5.12)$$

The definitions of y and \dot{x} , which are given in the relations (5.10) and (5.12), respectively, give us \dot{y} in the form

$$\dot{y} = -\frac{2\gamma}{x + y\frac{dx}{dy}} \cdot \left(\pm \sqrt{xy^2 - \kappa^2} \right) . \quad (5.13)$$

Now, let us express x as a function of y . Basic equation of this task is the relation (5.10). As for $\Lambda(x)$, we adopt the approximate form $\Lambda^a(x)$ given in the relation (4.51). For $\Lambda^a(x)$, the relation (5.10) is reduced to the form

$$CDy \cdot x^2 - (mCD - y(C + D)) \cdot x + (y - (kC + lD)) = 0 . \quad (5.14)$$

A solution of Eq.(5.14) is as follows:

$$x = \frac{m}{2y} - \frac{C + D}{2CD} + \frac{m}{2y} \sqrt{1 + 2Iy + J^2y^2} ,$$

$$I = \left(\frac{k-l}{m^2} \right) \left(\frac{C-D}{CD} \right) , \quad J^2 = \frac{1}{m^2} \left(\frac{C-D}{CD} \right)^2 . \quad \left(J^2 = \left(\frac{k+l}{k-l} \right)^2 I^2 \right) \quad (5.15)$$

In the case $C = D$, another solution becomes negative and we pick up only the solution (5.15). Next, we consider a possible approximation of $\sqrt{1 + 2Iy + J^2y^2}$, which, up to the term y^2 , is expanded for y :

$$\sqrt{1 + 2Iy + J^2y^2} = 1 + Iy + \frac{1}{2}(J^2 - I^2)y^2 . \quad (5.16)$$

Let the following inequality be permitted:

$$y \ll \left| \frac{2I}{J^2 - I^2} \right| . \quad (5.17)$$

Then, we are able to obtain the approximate form

$$\sqrt{1 + 2Iy + J^2y^2} = 1 + Iy = 1 + \left(\frac{k-l}{m^2} \right) \left(\frac{C-D}{CD} \right) y . \quad (5.18)$$

Later, we will discuss the condition, under which the inequality (5.17) is meaningful. Then, we have

$$x = \frac{m}{2y} - \frac{C + D}{2CD} + \frac{m}{2y} \left(1 + \left(\frac{k-l}{m^2} \right) \left(\frac{C-D}{CD} \right) y \right) = \frac{m}{y} - \frac{1}{B} , \quad (5.19)$$

$$\frac{1}{B} = \frac{1}{m} \left(\frac{k}{C} + \frac{l}{D} \right) . \quad (5.20)$$

With the use of the relation (5.19), we obtain

$$xy^2 - \kappa^2 = \left(\frac{m^2B}{4} - \kappa^2 \right) - \frac{1}{B} \left(y - \frac{mB}{2} \right)^2 , \quad (5.21a)$$

$$x + y\frac{dx}{dy} = -\frac{1}{B} . \quad (5.21b)$$

Therefore, \dot{y} shown in the relation (5.13) can be expressed as

$$\dot{y} = \pm 2\gamma B \sqrt{\left(\frac{m^2 B}{4} - \kappa^2\right) - \frac{1}{B} \left(y - \frac{mB}{2}\right)^2} . \quad (5.22)$$

By solving Eq.(5.22), y can be expressed as a function τ . The relation (5.22) can be rewritten to the form

$$\frac{1}{2}\dot{y}^2 + \frac{1}{2} \cdot 4\gamma^2 B \left(y - \frac{mB}{2}\right)^2 = 2\gamma^2 B \left[\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2\right] . \quad (5.23)$$

The relation (5.23) tells us that the present system is equivalent to a simple harmonic oscillator in the classical mechanics. Then, we have

$$y = \frac{mB}{2} + \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2} \cos(2\gamma\sqrt{B}\tau + \chi_0) . \quad (5.24)$$

Here, χ_0 is determined by the initial condition. The quantities x and $\Lambda^a(x)$ can be expressed in the following form:

$$x = \frac{\frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2} \cos(2\gamma\sqrt{B}\tau + \chi_0)}{\frac{mB}{2} + \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2} \cos(2\gamma\sqrt{B}\tau + \chi_0)} , \quad (5.25)$$

$$\Lambda^a(x) = \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2} \cos(2\gamma\sqrt{B}\tau + \chi_0) . \quad (5.26)$$

Thus, we could express x and $\Lambda^a(x)$ as functions of τ . Of course, it is a general solution, in which the initial and the boundary condition are not taken into account. In §6, we will discuss these conditions.

Finally, we will discuss the inequality (5.17), which leads us to the simple result shown in the relation (5.26). First, we note the maximum value of y , y_{\max} , which is expressed as

$$y_{\max} = mB . \quad (5.27)$$

The relation (5.27) is obtained under the condition ($\kappa = 0$, $\cos(2\gamma\sqrt{B}\tau + \chi_0) = 1$) in the relation (5.24). Therefore, we have

$$mB \ll \left| \frac{2I}{J^2 - I^2} \right| . \quad (5.28)$$

After rather lengthy consideration, the following inequality can be derived from the relation (5.28):

$$\zeta_0 \ll \zeta_0(j) , \quad (5.29)$$

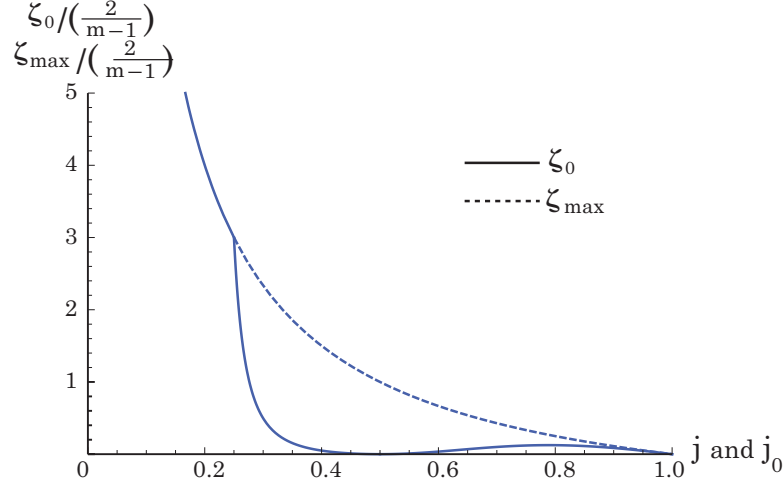


Fig. 2. For the comparison, $\zeta_0(j)$ in Eq.(5.30) and ζ_{\max} in Eq.(5.32) are depicted. The vertical axis represents $\zeta_0(j)$ and/or $\zeta_{\max}(j_0)$ in unit $2/(m-1)$.

$$(i) \quad \text{for } 0 < j \leq \frac{1}{4}, \quad \zeta_0(j) = \frac{2}{m-1} \cdot \frac{1-j}{j}, \quad (5.30a)$$

$$(ii) \quad \text{for } \frac{1}{4} < j < \frac{1}{2}, \quad \zeta_0(j) = \frac{2}{m-1} \cdot \frac{j(1-j)(1-2j)^2}{(1-6j+6j^2)^2}, \quad (5.30b)$$

$$(iii) \quad \text{for } \frac{1}{2} \leq j < 1, \quad \zeta_0(j) = \frac{2}{m-1} \cdot \frac{j(1-j)(1-2j)^2}{(1-2j+2j^2)^2}. \quad (5.30c)$$

Here, j denotes

$$j = \frac{l}{m}. \quad (0 < j < 1) \quad (5.31)$$

On the other hand, we note the inequality (4.48):

$$\zeta < \zeta_{\max}(j_0), \quad \zeta_{\max}(j_0) = \frac{2}{m-1} \cdot \frac{1-j_0}{j_0}. \quad (5.32)$$

Here, j_0 denotes

$$j_0 = \frac{l_0}{m}. \quad (0 < j_0 < 1) \quad (5.33)$$

Further, we note the relation (4.40), which can be expressed as

$$j_0 \geq j. \quad (5.34)$$

The inequality (5.29) and (5.32) suggest us the relation

$$\zeta_{\max}(j_0) \ll \zeta_0(j). \quad (5.35)$$

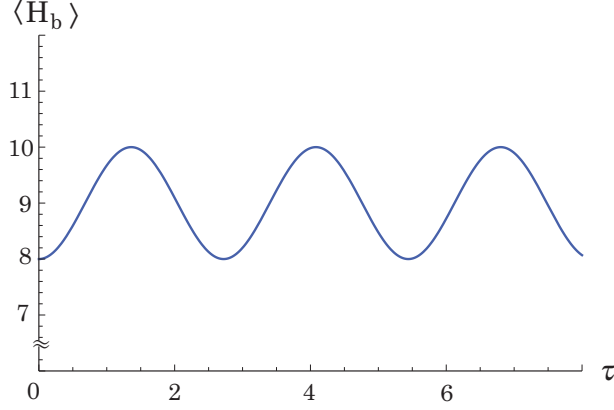


Fig. 3. The time-dependent energy for b -system is depicted as a function of time τ .

Figure 2 shows the behavior of ζ_0 and ζ_{\max} in unit $(2/(m-1))$. From the figure, we can learn the following points: (i) If $j \sim 0$ and $j_0 \sim 1$, the inequality (5.35) is sufficiently satisfied. (ii) If $1/4 \leq j < 1/2$ and $j_0 \sim 1$, the inequality (5.35) may be satisfied, but not so sufficient as the case (i). (iii) If $1/2 < j < 1$ and $1/2 < j_0 < 1$, the inequality (5.35) is not satisfied. The above summarize gives us the following conclusion: If l is rather far from l_0 ($l \ll l_0$), our approximation may be justified. Therefore, the case ($l = 1$, $l_0 = m - 1$) is the most reliable. This point has been already suggested in the previous section.

As an example of physical systems, let us consider b -system governed by the Hamiltonian (5.2) considered in this section. Figure 3 shows the energy expectation value for b -system as a function of time τ , which is depicted by using the approximation in Eq.(5.26). The parameters are taken as $t = 4$, $s = 3/2$, which leads to $t_m = 7$ and $m = 3$. Also, $l = 1$, $p = 1$, $q = 2$, $\gamma = 1$, $\omega = 1$ and $\kappa = 3/2$ are adopted and an initial condition, $\chi_0 = 0$, is given. It is seen that the energy flows into b -system from external environment and vice versa.

§6. Discussion

First of all, we will examine the formal result of the approximate solution (5.26) closely. For this aim, first, we consider the quantity B defined in the relation (5.20). With the use of the relations (4.45a) and (4.45b), B can be expressed in the form for the case ($k = m - 1$, $l = 1$) as follows:

$$B = q \left(1 - \sqrt{\frac{\zeta}{2}} \right) \left(1 + \frac{1}{m - \left(2 - \sqrt{\frac{2}{\zeta}} \right)} \right). \quad (6.1)$$

Since $m \geq 2$, B obeys the inequality

$$B \geq q \left(1 - \sqrt{\frac{\zeta}{2}} \right). \quad (6.2)$$

Table II. Examples for q and B .

q	B
1	1
2	$1 + \frac{1}{m}$
6	$(6 - \sqrt{5}) \left(1 + \frac{1}{m-2\left(1-\sqrt{\frac{3}{5}}\right)}\right) \approx 2.127 \left(1 + \frac{1}{m-0.451}\right)$
9	$3 \left(1 + \frac{1}{m-\frac{1}{2}}\right)$
13	$(13 - \sqrt{78}) \left(1 + \frac{1}{m-\left(2-\sqrt{\frac{6}{13}}\right)}\right) \approx 4.168 \left(1 + \frac{1}{m-1.321}\right)$

We are mostly interested in the case $p = 1$, i.e., $\zeta = 1 - 1/q$. Then, we have the relation

$$q \left(1 - \sqrt{\frac{\zeta}{2}}\right) - 1 = \frac{1}{\sqrt{2}} \frac{\sqrt{q-1}(q-2)}{\sqrt{2(q-1)} + \sqrt{q}} \geq 0. \quad (6.3)$$

The inequalities (6.2) and (6.3) lead us to

$$B \geq 1. \quad (6.4)$$

The examples are shown in Table II. Later at several places, we will use the inequality (6.4).

Now, we investigate the general solution (5.26). For this aim, we define two functions

$$\Lambda^a(\theta) = \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2} \cos \theta, \quad (6.5a)$$

$$y(\theta) = \frac{mB}{2} + \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2} \cos \theta. \quad (6.5b)$$

If θ is replaced with $(2\gamma\sqrt{B}\tau + \chi_0)$, $\Lambda^a(\theta)$ and $y(\theta)$ are reduced to the results (5.26) and (5.24). The present approximate result should obey the following boundary conditions:

$$(i) \quad y(\theta) \cdot \Lambda^a(\theta) - \kappa^2 \geq 0, \quad \text{i.e.,} \quad \left(\frac{mB}{2}\right)^2 - \left(\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2\right) \cos^2 \theta - \kappa^2 \geq 0, \quad (6.6a)$$

$$(ii) \quad \left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2 \geq 0, \quad (6.6b)$$

$$(iii) \quad \Lambda^a(\theta) \leq m, \quad \text{i.e.,} \quad \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2} \cos \theta \leq m. \quad (6.6c)$$

The condition (i) results from the relation (5.11), in which $(xy^2 - \kappa^2)$ can be expressed in the form $(y(\theta) \cdot \Lambda^a(\theta) - \kappa^2)$. The condition (ii) may be self-evident. The condition

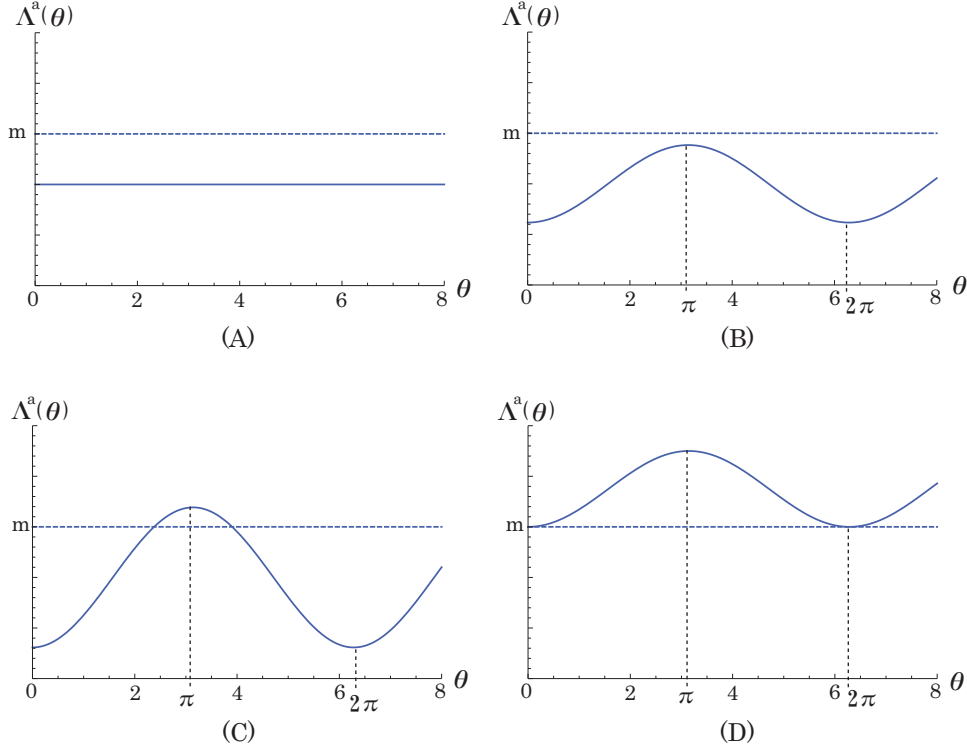


Fig. 4. Four cases under considerations are schematically depicted.

(iii) comes from the relations (4.52) and (4.53). On the basis of the conditions (i), (ii) and (iii), we examine our general solution (5.26).

Let us start in the condition (i). It is easily verified through the following inequality:

$$y(\theta) \cdot \Lambda^a(\theta) - \kappa^2 = \left(\left(\frac{mB}{2} \right)^2 - \left(\sqrt{B}\kappa \right)^2 \right) \sin^2 \theta + (B-1)\kappa^2 \geq 0. \quad (6.7)$$

Here, we used the conditions (6.4) and (6.6b). It is important to see that the condition (i) holds at any value of θ . It may be convenient to treat the condition (ii) by classifying it into two cases (a) and (b):

$$(a) \quad \left(\frac{mB}{2} \right)^2 - \left(\sqrt{B}\kappa \right)^2 = 0, \quad \text{i.e.,} \quad |\kappa| = \frac{m}{2} \sqrt{B}, \quad (6.8a)$$

$$(b) \quad \left(\frac{mB}{2} \right)^2 - \left(\sqrt{B}\kappa \right)^2 > 0, \quad \text{i.e.,} \quad |\kappa| < \frac{m}{2} \sqrt{B}. \quad (6.8b)$$

In the present treatment, κ is given as an initial condition and the case (a) gives us time-independent $\Lambda^a(=mB/2)$.

Consideration on the condition (iii) is rather lengthy and, then, only the results will be presented. In this case, it may be successful to consider the problem under

the following four cases depicted in Fig.4: The case (A) is nothing but the case (a) given in the relation (6.8a). Since $\Lambda^a(\tau) \leq m$, we have

$$1 \leq B \leq 2, \quad |\kappa| = \frac{m}{2}\sqrt{B}, \quad \Lambda^a(\tau) = \Lambda_A^a(\tau) = \frac{mB}{2}. \quad (6.9)$$

In the case (B), the following inequality holds:

$$\frac{mB}{2} + \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2} \leq m. \quad (6.10)$$

By solving the inequality (6.10), we have

$$1 \leq B < 2, \quad m\sqrt{1 - \frac{1}{B}} \leq |\kappa| < \frac{m}{2}\sqrt{B}. \quad (6.11)$$

Here, of course, we used the result of the case (b). Then, if at the initial time $\tau = 0$, $\theta = 0$ is chosen, i.e., $\chi_0 = 0$ in the general solution (5.26), we have

$$\Lambda^a(\tau) = \Lambda_B^a(\tau) = \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2} \cos(2\gamma\sqrt{B}\tau). \quad (6.12)$$

The case (C) satisfies the inequality

$$\frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2} < m < \frac{mB}{2} + \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2}. \quad (6.13)$$

In this case, we obtain

$$B > 1, \quad |\kappa| < m\sqrt{1 - \frac{1}{B}}. \quad (6.14)$$

We adopt the same initial condition as that in the above, $\theta = 0$, i.e., $\chi_0 = 0$ at $\tau = 0$. As is shown in Fig.4, there exists an angle θ_m and it is given in the form

$$\begin{aligned} \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2} \cos \theta_m &= m, \\ \text{i.e., } \cos \theta_m &= \frac{m\left(\frac{B}{2} - 1\right)}{\sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2}}, \quad (0 < \theta_m < \pi). \end{aligned} \quad (6.15)$$

At the time $\tau_m = \theta_m/(2\gamma\sqrt{B})$, $\Lambda^a(\theta_m) = m$ and in the interval $\tau = 0 \rightarrow \tau_m$, $\Lambda^a(\tau)$ can be expressed as

$$\Lambda^a(\tau) = \Lambda_0^a(\tau) = \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2} \cos(2\gamma\sqrt{B}\tau). \quad (6.16)$$

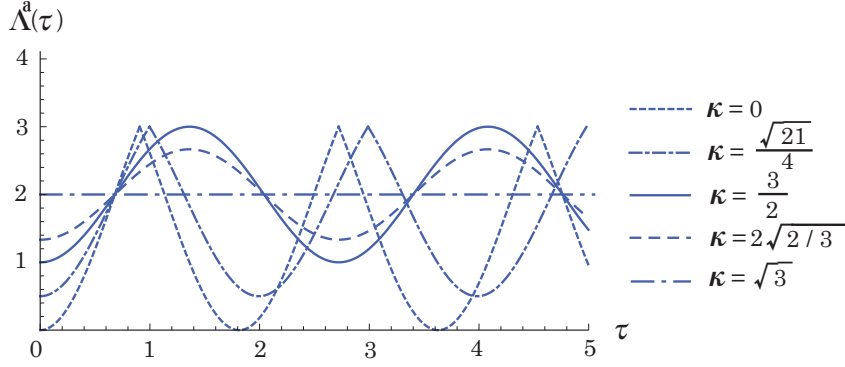


Fig. 5. The behavior of $\Lambda^a(\tau)$ for various values of κ is shown with the same parameters as those in Fig.3 except for κ .

However, after $\tau = \tau_m$, $\Lambda_0^a(\tau)$ cannot be adopted, because, if it is permitted, $\Lambda_0^a(\tau) > m$. Then, we define the following function:

$$\Lambda^a(\tau) = \Lambda_1^a(\tau) = \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2} \cos\left(2\gamma\sqrt{B}(\tau - 2\tau_m)\right). \quad (6.17)$$

The function $\Lambda_1^a(\tau)$ satisfies $\Lambda_1^a(\tau_m) = \Lambda_0^a(\tau_m) = m$ and in the interval $\tau = \tau_m \rightarrow 3\tau_m$, $\Lambda_1^a(\tau) < m$. Further, in the interval $\tau = 3\tau_m \rightarrow 5\tau_m$, we define $\Lambda_2^a(\tau)$ in the form

$$\Lambda^a(\tau) = \Lambda_2^a(\tau) = \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2} \cos\left(2\gamma\sqrt{B}(\tau - 4\tau_m)\right). \quad (6.18)$$

Certainly, $\Lambda_2^a(3\tau_m) = \Lambda_1^a(3\tau_m) = m$ and $\Lambda_2^a(\tau)$ is useful in the interval $\tau = 5\tau_m \rightarrow 7\tau_m$. By proceeding with this task, we arrive at the following solution:

$$\Lambda^a(\tau) = \Lambda_C^a(\tau) = \Lambda_n^a(\tau) = \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2} \cos\left(2\gamma\sqrt{B}(\tau - 2n\tau_m)\right),$$

for $(2n-1)\tau_m \leq \tau \leq (2n+1)\tau_m$. $(n = 0, 1, 2, 3 \dots)$ (6.19)

In the case (D), we have the relation

$$\frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - (\sqrt{B}\kappa)^2} = m. \quad (6.20)$$

Solution of this equation is given as

$$B > 2, \quad |\kappa| = m\sqrt{1 - \frac{1}{B}}, \quad \Lambda^a(\tau) = \Lambda_D^a(\tau) = m. \quad (6.21)$$

The case (D) is regarded as the limit $\theta_m \rightarrow 0$ in the case (C). The function $\Lambda_D^a(\tau)$ does not depend on τ , but its origin is different from that in the case (A). Figure 5 shows

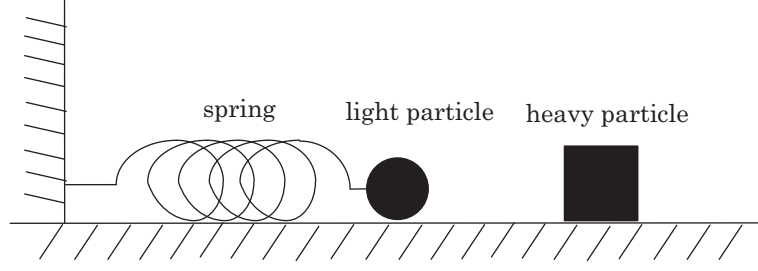


Fig. 6. The elastic collision of simply oscillating light particle with sufficiently heavy particle is illustrated.

the behavior of $\Lambda^a(\tau)$ for various values of κ . The same parameters as those used in Fig.3 are adopted except for κ which is a conserved quantity determined by the initial condition. Under this parameter set, we obtain $B = 4/3$ which is in the range $1 < B < 2$. Then, for various values of κ , the function $\Lambda^a(\tau)$ is turned into $\Lambda_C^a(\tau)$ or $\Lambda_B^a(\tau)$ or $\Lambda_A^a(\tau)$ according to Table III. For $\kappa = 0$ and $\kappa = \sqrt{21}/4 \approx 1.1456$, we take $\Lambda^a(\tau) = \Lambda_C^a(\tau)$. For $\kappa = 3/2$, we adopt $\Lambda^a(\tau) = \Lambda_C^a(\tau) = \Lambda_B^a(\tau)$. For $\kappa = 2\sqrt{2/3} \approx 1.633$, we chose $\Lambda^a(\tau) = \Lambda_B^a(\tau)$. Finally, for $\kappa = \sqrt{3} \approx 1.732$, we adopt $\Lambda^a(\tau) = \Lambda_B^a(\tau) = \Lambda_A^a(\tau)$.

In classical mechanics, we can find the same problem as that discussed in this section: elastic collision of simply oscillating light particle with sufficiently heavy particle, which is illustrated in Fig.6. The results obtained in the above are summarized in Table III.

In this paper, we proposed a new boson representation of the $su(2)$ -algebra. The basic idea comes from the pseudo $su(1, 1)$ -algebra in the Schwinger boson representation. In a certain sense, ours is on the opposite side of the Schwinger representation of the $su(2)$ -algebra. In next paper, Part II, we will prove that ours satisfies the $su(2)$ -algebra in the subspace (2.8) of the whole space (2.5) for the case $t_m = C_m + 1 - t$.

Table III.

B	κ	$\Lambda^a(\tau)$
$B = 1$	$0 \leq \kappa < \frac{m}{2}$	$\Lambda_B^a(\tau)$
	$ \kappa = \frac{m}{2}$	$\Lambda_A^a(\tau)$
$1 < B < 2$	$0 \leq \kappa < m\sqrt{1 - \frac{1}{B}}$	$\Lambda_C^a(\tau)$
	$m\sqrt{1 - \frac{1}{B}} \leq \kappa < \frac{m}{2}\sqrt{B}$	$\Lambda_B^a(\tau)$
	$ \kappa = \frac{m}{2}\sqrt{B}$	$\Lambda_A^a(\tau)$
$B = 2$	$0 \leq \kappa < \frac{m}{\sqrt{2}}$	$\Lambda_C^a(\tau)$
	$ \kappa = \frac{m}{\sqrt{2}}$	$\Lambda_A^a(\tau)$
$B > 2$	$0 \leq \kappa < m\sqrt{1 - \frac{1}{B}}$	$\Lambda_C^a(\tau)$
	$ \kappa = m\sqrt{1 - \frac{1}{B}}$	$\Lambda_D^a(\tau)$

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